W16. If $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma$ (Euler- Mascheroni constant), then compute the limits

(i) $\lim_{n\to\infty}(\gamma_n - \gamma)n$

(ii) $\lim_{n\to\infty}(\gamma_n\gamma_{n+1} - \gamma^2)n$

(iii) $\lim_{n\to\infty}(\gamma_n\gamma_{n+1}\gamma_{n+2} - \gamma^3)n$

(iv) $\lim_{n\to\infty}(\gamma_n\gamma_{n+1}\gamma_{n+2}\ldots\gamma_{n+m-1}\gamma_{n+m} - \gamma^{m+1})n$

D. M. Bătinetu-Giurgiu and Neculai Stanciu.

Solution by Arkady Alt, San Jose, California, USA.

Let $h_n := \sum_{k=1}^{n} \frac{1}{k}$. First we will prove that $h_n = \gamma + \ln n + \frac{1}{2n} + o\left(\frac{1}{n}\right) \Leftrightarrow \lim_{n\to\infty}(\gamma_n - \gamma)n = \frac{1}{2}$.

Let $a_n := h_n - \ln n - \frac{1}{2n}$ and $b_n := h_n - \ln n - \frac{1}{2n} + \frac{1}{n^2}, n \in \mathbb{N}$.

We will prove that $a_n < a_{n+1}, n \in \mathbb{N}$ and $b_n > b_{n+1}, n \in \mathbb{N}$.

Indeed, since $\frac{1}{n} - \frac{1}{2n^2} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}$ for any $n \in \mathbb{N}$ then

$a_{n+1} - a_n = h_{n+1} - \ln(n+1) - \frac{1}{2(n+1)} - h_n + \ln n + \frac{1}{2n} =$

$\frac{1}{2(n+1)} + \frac{1}{2n} - \ln\left(1 + \frac{1}{n}\right) > \frac{1}{2(n+1)} + \frac{1}{2n} - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right) = \frac{1}{6}\frac{n-2}{n^3(n+1)} \geq 0$

for any $n \geq 2$ and $a_2 = \frac{5}{4} - \ln 2 > \frac{1}{2} = a_1 \Leftrightarrow \frac{3}{4} > \ln 2$;

Also $b_n - b_{n+1} = h_n - \ln n - \frac{1}{2n} + \frac{1}{n^2} - \left(h_{n+1} - \ln(n+1) - \frac{1}{2(n+1)} + \frac{1}{(n+1)^2}\right) =$

$\ln\left(1 + \frac{1}{n}\right) - \frac{1}{2n} - \frac{1}{2(n+1)} + \frac{1}{n^2} - \frac{1}{(n+1)^2} >$

$\left(\frac{1}{n} - \frac{1}{2n^2}\right) - \frac{2n+1}{2n(n+1)} + \frac{2n+1}{n^2(n+1)^2} = \frac{3n+1}{2n^2(n+1)^2} > 0.$

Since $(a_n)_{n\geq1}$ increasing sequence, $(b_n)_{n\geq1}$ decreasing sequence and $a_n < b_n, n \in \mathbb{N}$ then both sequences converges to the same limit which is equal to $\gamma = \lim_{n\to\infty}(h_n - \ln n)$ and, therefore, $h_n - \ln n - \frac{1}{2n} < \gamma < h_n - \ln n - \frac{1}{2n} + \frac{1}{n^2} \Leftrightarrow$

$-\frac{1}{n^2} < h_n - \ln n - \gamma - \frac{1}{2n} < 0 \Leftrightarrow h_n - \ln n - \gamma - \frac{1}{2n} = o\left(\frac{1}{n}\right) \Leftrightarrow \gamma_n = \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right)$

(ii) Since $\gamma_n\gamma_{n+1} - \gamma^2 = \left(\gamma + \frac{1}{2n}\right)\left(\gamma + \frac{1}{2(n+1)}\right) - \gamma^2 + o\left(\frac{1}{n}\right) =$

$\frac{\gamma}{2n+2} + \frac{\gamma}{2n} + \frac{1}{2(2n^2+2n)} =$

$\gamma \cdot \frac{2n+1}{2n(n+1)} + o\left(\frac{1}{n}\right)$ then $\lim_{n\to\infty}(\gamma_n\gamma_{n+1} - \gamma^2)n = \lim_{n\to\infty}\left(\gamma \cdot \frac{2n+1}{2n(n+1)} + o\left(\frac{1}{n}\right)\right)n = \gamma.$

(iii) Since $\gamma_n\gamma_{n+1}\gamma_{n+2} - \gamma^3 = \left(\gamma + \frac{1}{2n}\right)\left(\gamma + \frac{1}{2(n+1)}\right)\left(\gamma + \frac{1}{2(n+2)}\right) - \gamma^3 + o\left(\frac{1}{n}\right) =$

$\frac{3\gamma}{4n(n+2)} + \frac{\gamma^2(3n^2+6n+2)}{2n(n+2)(n+1)} + o\left(\frac{1}{n}\right)$

then $\lim_{n\to\infty}(\gamma_n\gamma_{n+1}\gamma_{n+2} - \gamma^3)n = \frac{3\gamma^2}{2}.$

(iv)

Since $\frac{\gamma_n}{\gamma} = 1 + \frac{1}{2n\gamma} + o\left(\frac{1}{n}\right), n \in \mathbb{N}$ then for ane fixed $m \in \mathbb{N}$ we have

$$\prod_{k=1}^{m} \gamma_{n+k} - \gamma^m = \gamma^m\left(\prod_{k=n+1}^{n+m}\left(1 + \frac{1}{2k\gamma} + o\left(\frac{1}{n}\right)\right) - 1\right) = \gamma^m\left(\prod_{k=n+1}^{n+m}\left(1 + \frac{1}{2k\gamma}\right) - 1\right) + o\left(\frac{1}{n}\right).$$

Noting that $\prod_{k=n+1}^{n+m}\left(1 + \frac{1}{2k\gamma}\right) = 1 + \sum_{k=n+1}^{n+m}\frac{1}{2k\gamma} + o\left(\frac{1}{n}\right)$

(because $0 < \prod_{k=n+1}^{n+m}\left(1 + \frac{1}{2k\gamma}\right) - 1 - \sum_{k=n+1}^{n+m}\frac{1}{2k\gamma} < \sum_{i=2}^{m}\binom{m}{i}\frac{1}{2^i\gamma^i n^i} < \frac{1}{n^2}\sum_{i=2}^{m}\binom{m}{i}\frac{1}{2^i\gamma^i}$)

we obtain $\prod_{k=1}^{m} \gamma_{n+k} - \gamma^m = \gamma^m\left(\sum_{k=n+1}^{n+m}\frac{1}{2k\gamma} + o\left(\frac{1}{n}\right)\right) + o\left(\frac{1}{n}\right) = \frac{\gamma^{m-1}}{2}\sum_{k=n+1}^{n+m}\frac{1}{k} + o\left(\frac{1}{n}\right)$

and, therefore, $.\lim_{n\to\infty}\left(\prod_{k=1}^{m} \gamma_{n+k} - \gamma^m\right)n = \frac{\gamma^{m-1}}{2} \cdot \lim_{n\to\infty} n\sum_{k=n+1}^{n+m}\frac{1}{k} = \frac{\gamma^{m-1}}{2} \cdot m = \frac{m\gamma^{m-1}}{2}$

or by replacing (n, m) in $\lim_{n\to\infty}\left(\prod_{k=1}^{m} \gamma_{n+k} - \gamma^m\right)n = \frac{m\gamma^{m-1}}{2}$ with $(n-1, m+1)$

we obtain $\lim_{n\to\infty}\left(\prod_{k=1}^{m+1} \gamma_{n-1+k} - \gamma^{m+1}\right)(n-1) = \frac{(m+1)\gamma^m}{2} \Leftrightarrow$

$\lim_{n\to\infty}\left(\prod_{k=0}^{m} \gamma_{n+k} - \gamma^{m+1}\right)n = \frac{(m+1)\gamma^m}{2}$ (because $\lim_{n\to\infty}\left(\prod_{k=1}^{m+1} \gamma_{n-1+k} - \gamma^{m+1}\right) = 0$)